

T-partition systems and travel groupoids on a graph

JUNG RAE CHO JEONGMI PARK

Department of Mathematics, Pusan National University, Busan 609-735, Korea

jungcho@pusan.ac.kr

jm1015@pusan.ac.kr

YOSHIO SANO *

Division of Information Engineering

Faculty of Engineering, Information and Systems

University of Tsukuba, Ibaraki 305-8573, Japan

sano@cs.tsukuba.ac.jp

Abstract

The notion of travel groupoids was introduced by L. Nebeský in 2006 in connection with a study on geodetic graphs. A travel groupoid is a pair of a set V and a binary operation $*$ on V satisfying two axioms. For a travel groupoid, we can associate a graph. We say that a graph G has a travel groupoid if the graph associated with the travel groupoid is equal to G . Nebeský gave a characterization for finite graphs to have a travel groupoid.

In this paper, we introduce the notion of T-partition systems on a graph and give a characterization of travel groupoids on a graph in terms of T-partition systems.

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1 Introduction

All graphs in this paper are (finite or infinite) undirected graphs with no loops and no multiple edges.

A *groupoid* is the pair $(V, *)$ of a nonempty set V and a binary operation $*$ on V . The notion of travel groupoids was introduced by L. Nebeský [4] in 2006. First, let us recall the definition of travel groupoids.

A *travel groupoid* is a groupoid $(V, *)$ satisfying the following axioms (t1) and (t2):

$$(t1) \quad (u * v) * u = u \text{ (for all } u, v \in V),$$

$$(t2) \quad \text{if } (u * v) * v = u, \text{ then } u = v \text{ (for all } u, v \in V).$$

A travel groupoid is said to be *simple* if the following condition holds.

$$(t3) \quad \text{If } v * u \neq u, \text{ then } u * (v * u) = u * v \text{ (for any } u, v \in V).$$

A travel groupoid is said to be *smooth* if the following condition holds.

$$(t4) \quad \text{If } u * v = u * w, \text{ then } u * (v * w) = u * v \text{ (for any } u, v, w \in V).$$

A travel groupoid is said to be *semi-smooth* if the following condition holds.

$$(t5) \quad \text{If } u * v = u * w, \text{ then } u * (v * w) = u * v \text{ or } u * ((v * w) * w) = u * v \text{ (for any } u, v, w \in V).$$

Let $(V, *)$ be a travel groupoid, and let G be a graph. We say that $(V, *)$ *is on* G or that G *has* $(V, *)$ if $V(G) = V$ and $E(G) = \{\{u, v\} \mid u, v \in V, u \neq v, \text{ and } u * v = v\}$. Note that every travel groupoid is on exactly one graph.

In this paper, we introduce the notion of T-partition systems on a graph and give a characterization of travel groupoids on a graph. This paper is organized as follows: In Section 2, we define the right translation system of a groupoid and characterize travel groupoids in terms of the right translation systems of groupoids. Section 3 is the main part of this paper. We introduce T-partition systems on a graph and give a characterization of travel groupoids on a graph in terms of T-partition systems. In Section 4, we consider simple, smooth, and semi-smooth travel groupoids, and give characterizations for them.

2 The right translation system of a travel groupoid

Definition. Let $(V, *)$ be a groupoid. For $u, v \in V$, let $V_{u,v}^R$ be the set of elements whose right translations send the element u to the element v , i.e.,

$$V_{u,v} = V_{u,v}^R := \{w \in V \mid u * w = v\}.$$

Then we call the system $(V_{u,v} \mid (u, v) \in V \times V)$ the *right translation system* of the groupoid $(V, *)$. \square

Remark. For each element w in a groupoid $(V, *)$, the *right translation map* $R_w : V \rightarrow V$ is defined by $R_w(u) := u * w$ for $u \in V$. Then the set $V_{u,v}^R$ is given by $V_{u,v}^R = \{w \in V \mid R_w(u) = v\}$.

We can also consider the set $V_{u,v}^R$ defined above as the inverse image of $\{v\}$ through the left translation map $L_u : V \rightarrow V$ of the groupoid $(V, *)$ defined by $L_u(w) := u * w$ for $w \in V$, i.e., $V_{u,v}^R = L_u^{-1}(\{v\})$.

Lemma 2.1. *Let $(V, *)$ be a groupoid. Then, $(V, *)$ satisfies the condition (t1) if and only if the right translation system $(V_{u,v} \mid (u, v) \in V \times V)$ of $(V, *)$ satisfies the following property:*

(R1) *For any $u, v \in V$, if $V_{u,v} \neq \emptyset$, then $u \in V_{v,u}$.*

Proof. Let $(V, *)$ be a groupoid satisfying the condition (t1). Take any $u, v \in V$ such that $V_{u,v} \neq \emptyset$. Take an element $x \in V_{u,v}$. Then $u * x = v$. By (t1), we have $(u * x) * u = u$. Therefore, we have $v * u = u$ and so $u \in V_{v,u}$. Thus the property (R1) holds.

Suppose that the right translation system $(V_{u,v} \mid (u, v) \in V \times V)$ of a groupoid $(V, *)$ satisfies the property (R1). Take any $u, v \in V$. Then there exists a unique element $x \in V$ such that $v \in V_{u,x}$. Since $V_{u,x} \neq \emptyset$, it follows from the property (R1) that $u \in V_{x,u}$ and so $x * u = u$. Since $v \in V_{u,x}$, we have $u * v = x$. Therefore, we have $(u * v) * u = u$. Thus $(V, *)$ satisfies the condition (t1). \square

Lemma 2.2. *Let $(V, *)$ be a groupoid. Then, $(V, *)$ satisfies the condition (t2) if and only if the right translation system $(V_{u,v} \mid (u, v) \in V \times V)$ of $(V, *)$ satisfies the following property:*

(R2) *For any $u, v \in V$ with $u \neq v$, $V_{u,v} \cap V_{v,u} = \emptyset$.*

Proof. Let $(V, *)$ be a groupoid satisfying the condition (t2). Suppose that $V_{u,v} \cap V_{v,u} \neq \emptyset$ for some $u, v \in V$ with $u \neq v$. Take an element $x \in V_{u,v} \cap V_{v,u}$.

Then $u * x = v$ and $v * x = u$. Therefore, we have $(u * x) * x = u$, which is a contradiction to the property (t2). Thus the property (R2) holds.

Suppose that the right translation system $(V_{u,v} \mid (u,v) \in V \times V)$ of a groupoid $(V, *)$ satisfies the property (R2). Suppose that there exist $u, v \in V$ with $u \neq v$ such that $(u * v) * v = u$. Let $x := u * v$. Then $v \in V_{u,x}$. Since $(u * v) * v = u$, we have $x * v = u$ and so $v \in V_{x,u}$. Therefore $v \in V_{u,x} \cap V_{x,u}$, which is a contradiction to the property (R2). Thus $(V, *)$ satisfies the condition (t2). \square

The following gives a characterization of travel groupoids in terms of the right translation systems of groupoids.

Theorem 2.3. *Let $(V, *)$ be a groupoid. Then, $(V, *)$ is a travel groupoid if and only if the right translation system of $(V, *)$ satisfies the properties (R1) and (R2).*

Proof. The theorem follows from Lemmas 2.1 and 2.2. \square

3 T-partition systems on a graph

We introduce the notion of T-partition systems on a graph to characterize travel groupoids on a graph.

For a vertex u in a graph G , let $N_G[u]$ denote the closed neighborhood of u in G , i.e., $N_G[u] := \{u\} \cup \{v \in V \mid \{u, v\} \in E\}$.

Definition. Let $G = (V, E)$ be a graph. A *T-partition system* on G is a system

$$\mathcal{P} = (V_{u,v} \subseteq V \mid (u, v) \in V \times V)$$

satisfying the following conditions:

(P0) $\mathcal{P}_u := \{V_{u,v} \mid v \in N_G[u]\}$ is a partition of V (for any $u \in V$);

(P1a) $V_{u,u} = \{u\}$ (for any $u \in V$);

(P1b) $v \in V_{u,v} \iff \{u, v\} \in E$ (for any $u, v \in V$ with $u \neq v$);

(P1c) $V_{u,v} = \emptyset \iff \{u, v\} \notin E$ (for any $u, v \in V$ with $u \neq v$);

(P2) $V_{u,v} \cap V_{v,u} = \emptyset$ (for any $u, v \in V$ with $u \neq v$). \square

Remark. Let $\mathcal{P} = (V_{u,v} \mid (u, v) \in V \times V)$ be a T-partition system on a graph G . It follows from the conditions (P0) and (P1c) that, for any $u, v \in V$, there exists the unique vertex w such that $v \in V_{u,w}$.

Definition. Let $\mathcal{P} = (V_{u,v} \mid (u, v) \in V \times V)$ be a T-partition system on a graph G . For $u, v \in V$, let $f_u(v)$ be the unique vertex w such that $v \in V_{u,w}$. We define a binary operation $*$ on V by $u * v := f_u(v)$. We call $(V, *)$ the *groupoid associated with \mathcal{P}* . \square

Remark. It follows from definitions that the right translation system of the groupoid associated with a T-partition system \mathcal{P} on a graph G is the same as \mathcal{P} .

Lemma 3.1. *Let G be a graph, and let \mathcal{P} be a T-partition system on G . Then, the groupoid associated with \mathcal{P} is a travel groupoid on G .*

Proof. It follows from the properties (P1a), (P1b), and (P1c) in the definition a T-partition system on a graph that a T-partition system $\mathcal{P} = (V_{u,v} \mid (u, v) \in V \times V)$ satisfies the property (R1). Moreover, the condition (P2) is the same as the property (R2). By Theorem 2.3, $(V, *)$ is a travel groupoid on G .

Now we show that $(V, *)$ is on the graph G . Let $G_{(V,*)}$ be the graph which has $(V, *)$. We show that $G_{(V,*)} = G$. Take any edge $\{u, v\}$ in $G_{(V,*)}$. Then, we have $u * v = v$. Therefore $v \in V_{u,v}$. Thus $\{u, v\}$ is an edge in G , and so $E(G_{(V,*)}) \subseteq E(G)$. Take any edge $\{u, v\}$ in G . Then, we have $v \in V_{u,v}$. Therefore $u * v = v$. Thus $\{u, v\}$ is an edge in $G_{(V,*)}$, and so $E(G) \subseteq E(G_{(V,*)})$. Hence we have $G_{(V,*)} = G$. \square

Lemma 3.2. *Let G be a graph, and let $(V, *)$ be a travel groupoid on G . Then, the right translation system of $(V, *)$ is a T-partition system on G .*

Proof. Let $(V_{u,v} \mid (u, v) \in V \times V)$ be the right translation system of a travel groupoid $(V, *)$ on a graph G .

Fix any $u \in V$. Since $u * w$ is in $N_G[u]$ for any $w \in V$, we have $\bigcup_{v \in N_G[u]} V_{u,v} = V$. Suppose that $V_{u,x} \cap V_{u,y} \neq \emptyset$ for some $x, y \in N_G[u]$ with $x \neq y$. Take $z \in V_{u,x} \cap V_{u,y}$. Then $u * z = x$ and $u * z = y$. Therefore we have $x = y$, which is a contradiction to the assumption $x \neq y$. Thus $V_{u,x} \cap V_{u,y} = \emptyset$ for any $x, y \in N_G[u]$ with $x \neq y$. Therefore $\{V_{u,v} \mid v \in N_G[u]\}$ is a partition of V . Thus the condition (P0) holds.

By [4, Proposition 2 (2)], $u * v = u$ if and only if $u = v$. Therefore $V_{u,u} = \{u\}$ and thus the condition (P1a) holds.

If u and v are adjacent in G , then we have $u * v = v$, and so $v \in V_{u,v}$. If $v \in V_{u,v}$, then we have $u * v = v$, and so $\{u, v\}$ is an edge of G . Thus the condition (P1b) holds.

Since $u * w$ is a neighbor of u in G for any $w \in V \setminus \{u\}$, if u and v are not adjacent in G , then $V_{u,v} = \emptyset$. If $V_{u,v} = \emptyset$, then we have $u * v \neq v$, and so u and v are not adjacent in G . Thus the condition (P1c) holds.

Since $(V, *)$ satisfies the condition (t2), it follows from Lemma 2.2 that the condition (P2) holds.

Hence the right translation system of $(V, *)$ is a T-partition system on G . \square

The following gives a characterization of travel groupoids on a graph in terms of T-partition systems on the graph.

Theorem 3.3. *Let G be a graph. Then, there exists a one-to-one correspondence between the set of all travel groupoids on G and the set of all T-partition systems on G .*

Proof. Let $V := V(G)$. Let $\text{TG}(G)$ denote the set of all travel groupoids on G and let $\text{TPS}(G)$ denote the set of all T-partition systems on G .

We define a map $\Phi : \text{TPS}(G) \rightarrow \text{TG}(G)$ as follows: For $\mathcal{P} \in \text{TPS}(G)$, let $\Phi(\mathcal{P})$ be the groupoid associated with \mathcal{P} . By Lemma 3.1, $\Phi(\mathcal{P})$ is a travel groupoid on G .

We define a map $\Psi : \text{TG}(G) \rightarrow \text{TPS}(G)$ as follows: For $(V, *) \in \text{TG}(G)$, let $\Psi((V, *))$ be the right translation system of $(V, *)$. By Lemma 3.2, $\Psi((V, *))$ is a T-partition system on G .

Then, we can check that $\Psi(\Phi(\mathcal{P})) = \mathcal{P}$ holds for any $\mathcal{P} \in \text{TPS}(G)$ and that $\Phi(\Psi((V, *))) = (V, *)$ holds for any $(V, *) \in \text{TG}(G)$. Hence the map Φ is a one-to-one correspondence between the sets $\text{TPS}(G)$ and $\text{TG}(G)$. \square

Example 3.4. Let $G = (V, E)$ be the graph defined by $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\}$. Let $(V, *)$ be the groupoid defined by

$$\begin{aligned} a * a &= a, & a * b &= b, & a * c &= d, & a * d &= d, \\ b * a &= a, & b * b &= b, & b * c &= c, & b * d &= a, \\ c * a &= b, & c * b &= b, & c * c &= c, & c * d &= d, \\ d * a &= a, & d * b &= c, & d * c &= c, & d * d &= d. \end{aligned}$$

Then $(V, *)$ is a travel groupoid on the graph G .

Let $\mathcal{P} = (V_{u,v} \mid (u, v) \in V \times V)$ be the system of vertex subsets defined by

$$\begin{aligned} V_{a,a} &= \{a\}, & V_{a,b} &= \{b\}, & V_{a,c} &= \emptyset, & V_{a,d} &= \{c, d\}, \\ V_{b,a} &= \{a, d\}, & V_{b,b} &= \{b\}, & V_{b,c} &= \{c\}, & V_{b,d} &= \emptyset, \\ V_{c,a} &= \emptyset, & V_{c,b} &= \{a, b\}, & V_{c,c} &= \{c\}, & V_{c,d} &= \{d\}, \\ V_{d,a} &= \{a\}, & V_{d,b} &= \emptyset, & V_{d,c} &= \{b, c\}, & V_{d,d} &= \{d\}. \end{aligned}$$

Then \mathcal{P} is a T-partition system on G .

Now we can see that the right translation system of $(V, *)$ is \mathcal{P} and that the groupoid associated with \mathcal{P} is $(V, *)$. \square

4 Simple, smooth, and semi-smooth systems

In this section, we give characterizations of simple, smooth, and semi-smooth travel groupoids on a graph in terms of T-partition systems on the graph.

4.1 Simple T-partition systems

Proposition 4.1. *Let $(V, *)$ be a travel groupoid. For $u, v \in V$, let*

$$V_{u,v} := \{w \in V \mid u * w = v\}.$$

Then, the following conditions are equivalent:

- (a) $(V, *)$ is simple;
- (b) For $u, v, x, y \in V$, if $u \in V_{v,x}$, $v \in V_{u,y}$, $u \neq x$, and $v \neq y$, then $x \in V_{u,y}$ and $y \in V_{v,x}$.

Proof. First, we show that (a) implies (b). Take any $u, v, x, y \in V$ such that $u \in V_{v,x}$, $v \in V_{u,y}$, $u \neq x$, $v \neq y$. Then $v * u = x \neq u$ and $u * v = y \neq v$. Since $(V, *)$ is simple by (a), $v * u \neq u$ implies $u * (v * u) = u * v$ and $u * v \neq v$ implies $v * (u * v) = v * u$. Therefore, we have $u * x = y$ and $v * y = x$. Thus $x \in V_{u,y}$ and $y \in V_{v,x}$.

Second, we show that (b) implies (a). Take any $u, v \in V$ such that $v * u \neq u$. Note that $v * u \neq u$ implies $u * v \neq v$ (cf. [4, Proposition 2 (1)]). Let $x := v * u$. Then $u \in V_{v,x}$ and $u \neq x$. Let $y := u * v$. Then $v \in V_{u,y}$ and $v \neq y$. By (b), we have $x \in V_{u,y}$ and $y \in V_{v,x}$. Therefore $u * x = y$, that is, $u * (v * u) = u * v$. Thus $(V, *)$ is simple. \square

Definition. A T-partition system $(V_{u,v} \mid (u, v) \in V \times V)$ on a graph G is said to be *simple* if the following condition holds:

- (R3) For $u, v, x, y \in V$, if $u \in V_{v,x}$, $v \in V_{u,y}$, $u \neq x$, and $v \neq y$, then $x \in V_{u,y}$ and $y \in V_{v,x}$. \square

Lemma 4.2. *Let G be a graph, and let \mathcal{P} be a simple T-partition system on G . Then, the groupoid associated with \mathcal{P} is a simple travel groupoid on G .*

Proof. The lemma follows from Lemma 3.1 and Proposition 4.1. \square

Lemma 4.3. *Let G be a graph, and let $(V, *)$ be a simple travel groupoid on G . Then, the right translation system of $(V, *)$ is a simple T-partition system on G .*

Proof. The lemma follows from Lemma 3.2 and Proposition 4.1. \square

Theorem 4.4. *Let G be a graph. Then, there exists a one-to-one correspondence between the set of all smooth travel groupoids on G and the set of all smooth T -partition systems on G .*

Proof. The theorem follows from Theorem 3.3 and Lemmas 4.2 and 4.3. \square

4.2 Smooth T -partition systems

Proposition 4.5. *Let $(V, *)$ be a travel groupoid. For $u, v \in V$, let*

$$V_{u,v} := \{w \in V \mid u * w = v\}.$$

Then, the following conditions are equivalent:

- (a) $(V, *)$ is smooth;
- (b) For $u, v, x, y \in V$, if $x, y \in V_{u,v}$, then $x * y \in V_{u,v}$;
- (c) For $u, v, x, y, z \in V$, if $x, y \in V_{u,v}$ and $x \in V_{y,z}$, then $z \in V_{u,v}$.

Proof. First, we show that (a) implies (b). Take any $u, v, x, y \in V$ such that $x, y \in V_{u,v}$. Then $u * x = v$ and $u * y = v$, so $u * x = u * y$. Since $(V, *)$ is smooth by (a), $u * x = u * y$ implies $u * (x * y) = u * x = v$. Thus $x * y \in V_{u,v}$.

Second, we show that (b) implies (c). Take any $u, v, x, y, z \in V$ such that $x, y \in V_{u,v}$ and $y \in V_{x,z}$. Then $x * y = z$. By (b), we have $x * y \in V_{u,v}$. Thus $z \in V_{u,v}$.

Third, we show that (c) implies (a). Take any $u, x, y \in V$ such that $u * x = u * y$. Let $v := u * x = u * y$. Then $x, y \in V_{u,v}$. Let $z := x * y$. Then $y \in V_{x,z}$. By (c), we have $z \in V_{u,v}$. Therefore $u * z = v$, that is, $u * (x * y) = u * x$. Thus $(V, *)$ is smooth. \square

Definition. A T -partition system $(V_{u,v} \mid (u, v) \in V \times V)$ on a graph G is said to be *smooth* if the following condition holds:

- (R4) For $u, v, x, y, z \in V$, if $x, y \in V_{u,v}$ and $x \in V_{y,z}$, then $z \in V_{u,v}$. \square

Lemma 4.6. *Let G be a graph, and let \mathcal{P} be a smooth T -partition system on G . Then, the groupoid associated with \mathcal{P} is a smooth travel groupoid on G .*

Proof. The lemma follows from Lemma 3.1 and Proposition 4.5. \square

Lemma 4.7. *Let G be a graph, and let $(V, *)$ be a smooth travel groupoid on G . Then, the right translation system of $(V, *)$ is a smooth T -partition system on G .*

Proof. The lemma follows from Lemma 3.2 and Proposition 4.5. \square

Theorem 4.8. *Let G be a graph. Then, there exists a one-to-one correspondence between the set of all smooth travel groupoids on G and the set of all smooth T-partition systems on G .*

Proof. The theorem follows from Theorem 3.3 and Lemmas 4.6 and 4.7. \square

Remark. Matsumoto and Mizusawa [3] gave an algorithmic way to construct a smooth travel groupoid on a finite connected graph. \square

4.3 Semi-smooth T-partition systems

Proposition 4.9. *Let $(V, *)$ be a travel groupoid. For $u, v \in V$, let*

$$V_{u,v} := \{w \in V \mid u * w = v\}.$$

Then, the following conditions are equivalent:

- (a) $(V, *)$ is semi-smooth;
- (b) For $u, v, x, y \in V$, if $x, y \in V_{u,v}$, then $x * y \in V_{u,v}$ or $x *^2 y \in V_{u,v}$;
- (c) For $u, v, x, y, z, w \in V$, if $x, y \in V_{u,v}$ and $x \in V_{y,z} \cap V_{z,w}$, then $z \in V_{u,v}$ or $w \in V_{u,v}$,

where $x *^2 y := (x * y) * y$.

Proof. First, we show that (a) implies (b). Take any $u, v, x, y \in V$ such that $x, y \in V_{u,v}$. Then $u * x = v$ and $u * y = v$, so $u * x = u * y$. Since $(V, *)$ is semi-smooth by (a), $u * x = u * y$ implies $u * (x * y) = u * x = v$ or $u * ((x * y) * y) = u * x = v$. Thus $x * y \in V_{u,v}$ or $x *^2 y = (x * y) * y \in V_{u,v}$.

Second, we show that (b) implies (c). Take any $u, v, x, y, z, w \in V$ such that $x, y \in V_{u,v}$ and $y \in V_{x,z} \cap V_{z,w}$. Then $x * y = z$ and $z * y = w$. Therefore $w = (x * y) * y = x *^2 y$. By (b), we have $x * y \in V_{u,v}$ or $x *^2 y \in V_{u,v}$. Thus $z \in V_{u,v}$ or $w \in V_{u,v}$.

Third, we show that (c) implies (a). Take any $u, x, y \in V$ such that $u * x = u * y$. Let $v := u * x = u * y$. Then $x, y \in V_{u,v}$. Let $z := x * y$ and $w := z * y = (x * y) * y$. Then $y \in V_{x,z} \cap V_{z,w}$. By (c), we have $z \in V_{u,v}$ or $w \in V_{u,v}$. Therefore $u * z = v$ or $u * w = v$, that is, $u * (x * y) = u * x$ or $u * ((x * y) * y) = u * x$. Thus $(V, *)$ is semi-smooth. \square

Definition. A T-partition system $(V_{u,v} \mid (u, v) \in V \times V)$ on a graph G is said to be *semi-smooth* if the following condition holds:

(R5) For $u, v, x, y, z, w \in V$, if $x, y \in V_{u,v}$ and $x \in V_{y,z} \cap V_{z,w}$, then $z \in V_{u,v}$ or $w \in V_{u,v}$. \square

Lemma 4.10. *Let G be a graph, and let \mathcal{P} be a semi-smooth T -partition system on G . Then, the groupoid associated with \mathcal{P} is a semi-smooth travel groupoid on G .*

Proof. The lemma follows from Lemma 3.1 and Proposition 4.9. \square

Lemma 4.11. *Let G be a graph, and let $(V, *)$ be a semi-smooth travel groupoid on G . Then, the right translation system of $(V, *)$ is a semi-smooth T -partition system on G .*

Proof. The lemma follows from Lemma 3.2 and Proposition 4.9. \square

Theorem 4.12. *Let G be a graph. Then, there exists a one-to-one correspondence between the set of all semi-smooth travel groupoids on G and the set of all semi-smooth T -partition systems on G .*

Proof. The theorem follows from Theorem 3.3 and Lemmas 4.10 and 4.11. \square

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